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The regulation of mechanical drive systems

R. Whalley *, M. Ebrahimi

Department of Mechanical and Manufacturing Engineering, University of Bradford, Bradford, West Yorkshire BD7 1DP, UK

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Abstract

The regulation of the performance of mechanical drive systems using output measurements alone is considered. Feedback control is employed to achieve specified decay and frequency characteristics. Illustrative results are outlined and the system responses are computed confirming the theoretical predictions thereby. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Matrix impedance models; Control; Frequency; Decay rates

Nomenclature

J	mass or inertia matrix $(m \times m)$
K_1	stiffness matrix $(m \times m)$
$T(t)$	input vector $(m \times 1)$
$\theta(t)$	displacement vector $(m \times 1)$
$\omega(t)$	velocity vector $(m \times 1)$
J_k	elements of J $(1 \leqslant k \leqslant m)$
k_{ij}	elements of K_1 $(1 \leqslant i, j \leqslant m)$
K	feedback vector $(m \times 1)$
H	measurement vector $(1 \times m)$
C	damping or frictional matrix $(m \times m)$
A_0	I_m identity matrix $(m \times m)$
A_1	$J_1^{-1}C$ normalised damping matrix $(m \times m)$
A_2	$J^{-1}K_1$ normalised stiffness matrix $(m \times m)$
c_j	elements of C $(1 \leqslant j \leqslant m)$
v	damping ratio (scalar)
ω_n	natural frequency (scalar)
s	Laplace variable $\sigma + i\omega$ (scalar)
$\xi(s)$	$s^2 + 2v\omega_n s$ (function)
$T(s), \theta(s), \omega(s)$	Laplace transforms of $T(t)$, $\theta(t)$, $\omega(t)$, respectively

* Corresponding author. Tel.: +44-01274-234539; fax: +44-01274-234525.
E-mail address: amst@bradford.ac.uk (R. Whalley).

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D	Heavyside's operator (operator)
U	eigenvector matrix ($m \times m$)
Λ	diagonal eigenvalue matrix ($m \times m$)
$-\lambda_j$	eigenvalues ($1 \leqslant j \leqslant m$)
$\langle a, b \rangle$	inner product of vectors a, b (scalar)
$a\rangle\langle b$	outer product of vectors a, b ($m \times m$)
Π	product operator (operator)
V	U^{-1} = left-hand eigenvector matrix ($m \times m$)
$E_m(\lambda)$	elementary function matrix ($m \times m$)
γ_j	natural frequencies ($1 \leqslant j \leqslant m$)
$W_m(\gamma)$	Wronskian matrix ($m \times m$)
$P(\gamma, \lambda)$	product vector ($m \times 1$)
$D_m(\gamma)$	product vector ($m \times 1$)

1. Introduction

Mechanical drive systems which can be described in terms of components comprising pointwise mass/inertia, frictional and stiffness properties may be represented by sets of simultaneous differential equations. For small perturbations these equations are linear and arise in Lagrangian form enabling Laplace transformation to the complex frequency domain, as shown in [1].

In matrix form, with zero initial conditions simple matrix quadratic, algebraic representations can be constructed, as shown in [2], wherein the principal coupling usually occurs in the steady state or stiffness matrix owing to the series, interconnected nature of the process. Moreover, due to this topology the mass/inertia matrix is often diagonal and the structure and coefficient values in the frictional matrix are usually known with least confidence.

Frictional dissipation in mechanical systems is invariably small owing to the use of ball and roller bearings, to reduce energy consumption and the low internal damping and high stiffness of metallic materials, such as steel, bronze, etc., which are commonly used. This and the use of modern lubricants enables highly efficient, steady state, operational effectiveness to be achieved though sustained oscillatory transient conditions, in accordance with the analyses given by Prentis and Leckie [3], are also encountered.

As well as inducing greater wear and noise, oscillatory behaviour creates fatigue in the highest stressed components and increases machine maintenance costs reducing reliability thereby. This is instrumental in interrupting continuous duty cycles and attracting additional production costs. The answer to the problem is provided by introducing additional energy dissipation confining thereby the amplitude of vibrations by way of mechanical or electrical dampers or feedback. These devices take a variety of forms such as shock absorbers, torsional dampers, friction and eddy current brakes and tachogenerator feedback all which are often incorporated as part of the initial design in anticipation of the final requirement. Electrical dampers have the additional flexibility of adjustment ease and unlike mechanical dampers do not significantly affect the mass or inertia of the system.

Moreover, when feedback is to be deployed for speed or position control, the servo motors employed are often supplied with integral tachogenerators for regulation purposes in anticipation of this requirement. Equally, in high speed rotational systems, for example, it is very easy to apply this kind of electrical or mechanical damping at various strategic locations to minimise oscillatory behaviour achieving thereby specified performance aims.

In exacting situations there are often particular decay rates or frequencies of oscillation which are acceptable or otherwise an over damped characteristic may be all that can be tolerated. To accommodate these requirements "testing and tuning" procedures are often implemented before deployment enabling the open or closed loop system elements to be adjusted accordingly.

To incorporate this flexibility the design must provide sufficient damping capacity, of the required rating, with suitable direct or feedback adjustment to achieve acceptable operational conditions.

Moreover, typical results which should be obtainable with specified settings in terms of decay rates, frequencies of oscillation and overshoot envelopes, etc., should also be available for purposes of comparison.

There is, however, much more that could be achieved with single input, multiple output mechanical systems such as machinery and servo mechanism drives or with electrical circuits when energy dissipation elements are used for regulation purposes. Under these conditions, with the aid of a simple feedback structure, precise damped frequencies of oscillation and decay rates can be effected.

In this exposition realisations with lumped mass/inertia matrix models where there is little natural damping, as with typical mechanical, series connected configurations, are considered as in [4]. Single input–multiple output linear descriptions are employed and there is no restriction on the structure of the stiffness matrix though this usually occurs, owing to series coupling conditions in physical systems, in symmetrical form, as shown in [5].

The objective proposed is to introduce, via feedback regulation, control over the decay rate and damped natural frequencies of the transient response of the system model. Algorithms enabling the synthesis of suitable feedback control for this purpose will be derived herein employing output feedback alone and simple proportional and/or derivative control action.

The procedures presented provide many of the advantages claimed for arbitrary pole assignment first proposed in [6], which is outlined in [7]. However, the observers used to generate the state vector are not necessary here and the control law algorithm is simpler than that proposed by Simon and Mitter [8]. The derivation is also elegant avoiding the use of complex algebra, state space models or transfer functions as in [9,11,12].

The use of proportional and/or derivative feedback is also proposed enabling additional energy dissipation to be provided. This influences both the damping ratio and damped natural frequency which may be adjusted to provide specific closed loop pole locations using the impedance matrix description of the system alone.

It is true that the compromise to be made requires the selection of two different coordinates other than the real and complex locations of the system poles. The choice offering attractive advantages appears to be the specification of the decay rate and the natural frequencies of oscillation, as will be shown.

2. Model and feedback structure

In accordance with the model description given earlier simple conservative models, as shown in [10], take the form

$$J\ddot{\theta} + K_1\theta = T_i, \tag{2.1}$$

where in Eq. (2.1) for series coupled configurations, such as that shown in Fig. 1, the general inertia matrix, for purposes of illustration, is

$$J = \mathrm{Diag}(J_1, J_2, \ldots, J_m)$$

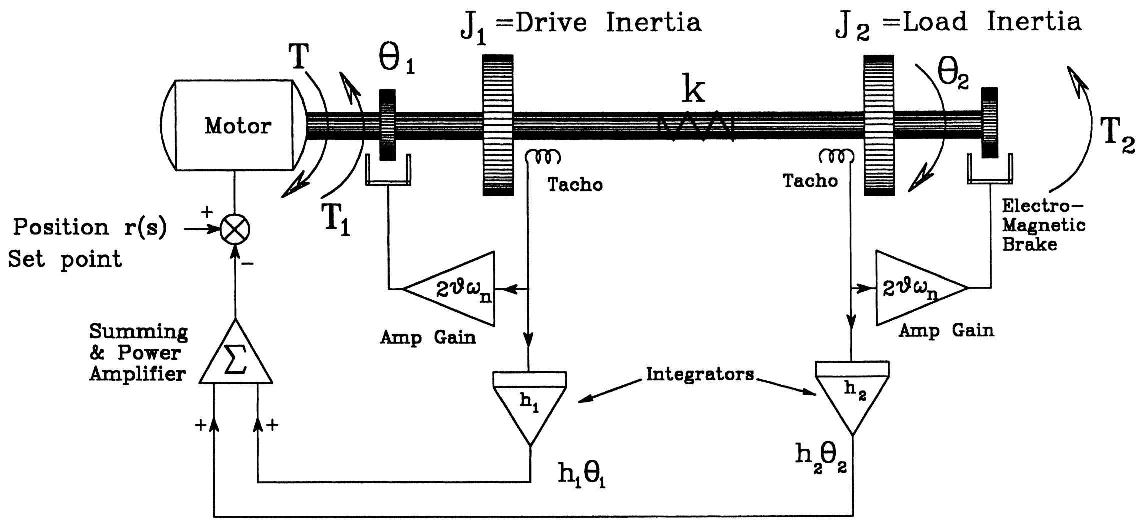

Fig. 1. Closed loop drive.

though any structure which is non-singular could be selected for J and

$K_1 =$ a constant $m \times m$ stiffness matrix.

The displacement vector is

$$\theta = (\theta_1, \theta_2, \ldots, \theta_m)^{\mathrm{t}},$$

and

$$T_i = (T, 0, \ldots, 0)^{\mathrm{t}} - (T_1, T_2, \ldots, T_m)^{\mathrm{t}},$$

where

$(T_1, 0, \ldots, 0)^{\mathrm{t}}$ is the driving torque vector

and

$(T_1, T_2, \ldots, T_m)^{\mathrm{t}}$ is the retarding torque vector.

Inverting the inertia matrix in Eq. (2.1) yields, following pre-multiplication:

$$\ddot{\theta} + A_2\theta = (T/J_1, 0, 0, \ldots, 0) - (T_1/J_1, T_2/J_2, \ldots, T_m/J_m), \quad (2.2)$$

where in Eq. (2.2)

$$A_2 = J^{-1}K_1.$$

If feedback is now introduced so that

$$T_i = K(r - H\theta) - J^{-1}C\omega, \quad (2.3)$$

where in Eq. (2.3)

$K = (k/J_1, 0, \ldots, 0)^{\mathrm{t}} \quad (m \times 1)$ feedback vector,

$r =$ reference input (scalar),

$H = (h_1, h_2, \ldots, h_m) \quad (1 \times m)$ measurement vector,

$C = \mathrm{Diag}(c_1, c_2, \ldots, c_m)$,

$\theta = (\theta_1, \theta_2, \ldots, \theta_m)^{\mathrm{t}} =$ position vector,

and

$\omega = (\dot{\theta}_1, \dot{\theta}_2, \ldots, \dot{\theta}_m)^{\mathrm{t}} =$ velocity vector.

To enable the required adjustment of the decay rate of the transient some form of damping must be applied so that the feedback $C\omega$ is available in Eq. (2.3). This could be provided by tacho-generator feedback, eddy current braking or dynomometer dissipation, etc.

Consequently, with this provision Eq. (2.2) becomes

$$I_m\ddot{\theta} + J_1^{-1}C\dot{\theta} + A_2\theta = K(r - H\theta), \tag{2.4}$$

which following Laplace transformation with zero initial conditions result in Eq. (2.4) becoming the standard impedance matrix quadratic

$$(A_0 s^2 + A_1 s + A_2)\theta(s) = K(r - H\theta(s)), \tag{2.5}$$

where in Eq. (2.5)

$$A_0 = I_m, \quad A_1 = J^{-1}C, \quad A_2 = J^{-1}K_1$$

and

$$\theta(s) = L\{\theta_0(t)\}.$$

If the decay rate is now set to the required value by adjusting $c_1, c_2, \ldots, c_m$ in

$$J^{-1}C = (c_1/J_1, c_2/J_2, \ldots, c_m/J_m)$$

such that c_k/J_k becomes approximately equal $1 \leqslant k \leqslant m$, as in practical terms the decay rate would have to be in series coupled systems, then Eq. (2.5) could be written in the form

$$(\xi(s)I_m + A_2)\theta(s) = K(r - H\theta(s)), \tag{2.6}$$

where in Eq. (2.6)

$$\xi(s) = s^2 + 2v\omega_{\mathrm{n}}s \tag{2.7}$$

and in Eq. (2.7) v is the damping ratio and ω_{n} the natural frequency.

3. Analysis

Returning now to Eq. (2.6) enables this expression to be arranged in the form

$$(\xi(s)I_m + A_2 + KH)\theta(s) = Kr, \tag{3.1}$$

so that upon inverting the LHS of Eq. (3.1) provides the solution

$$\theta(s) = (\xi(s)I_m + A_2 + KH)^{-1}Kr(s), \tag{3.2}$$

where in Eq. (3.2), as shown in [13]:

$$(\xi(s)I_m + A_2 + KH)^{-1} = \mathrm{Adj}(\xi(s)I_m + A_2 + KH)/\det(\xi(s)I_m + A_2 + KH). \tag{3.3}$$

The characteristic equation for the system model, from Eq. (3.3) is therefore given by the determinantal equation

$$\det(\xi(s)I_m + A_2 + KH) = 0. \tag{3.4}$$

Eq. (3.4) can in fact be arranged as

$$\det(\xi(s)I_m + A_2)(I_m + (\xi(s)I_m + A_2)^{-1}KH) \tag{3.5}$$

and since in Eq. (3.5)

$$(\xi(s)I_m + A_2)^{-1} = \mathrm{Adj}(\xi(s)I_m + A_2)/\det(\xi(s)I_m + A_2), \tag{3.6}$$

then from Eqs. (3.5) and (3.6) it is evident that

$$\det(\delta(s)I_m + \mathrm{Adj}(\xi(s)I_m + A_2)KH) = 0 \tag{3.7}$$

will provide the characteristic zeros for the system model where in Eq. (3.7)

$$\delta(s) = \det(\xi(s)I_m + A_2). \tag{3.8}$$

Eq. (3.7) can also be factorised, as shown in [14], such that it becomes

$$\det(\delta(s)I_m + \mathrm{Adj}(U(\xi(s)I_m + \Lambda)U^{-1})KH) = 0, \tag{3.9}$$

so that upon re-arranging Eq. (3.9) may be written as

$$\det(\delta(s)I_m + U\,\mathrm{Adj}(\xi(s)I_m + \Lambda)U^{-1}KH) = 0. \tag{3.10}$$

In Eq. (3.10) the adjoint matrix is easily obtained from the diagonal polynomial matrix by replacing each non-diagonal element by the product of those elements remaining. In Eq. (3.10)

Λ is the $(m \times m)$ matrix, $\mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$

and

U is the $(m \times m)$ eigenvector matrix

both of which are associated with A_2.

4. The determinantal equation

The determinantal or characteristic equation given by Eq. (3.10) is of a special form owing to the multiplication of the rank one gain matrix KH which is the outer product

$K\rangle\langle H.$

As shown in [15] the determinant of Eq. (3.10) can be simply evaluated from

$$\delta(s) + \langle HU\,\mathrm{Adj}(\xi(s)I_m + \Lambda)U^{-1}K\rangle = 0, \tag{4.1}$$

where Eq. (4.1) may be written as

$$\delta(s) + \left\langle HU\,\mathrm{Diag}\left(\prod_{\substack{j=1\\j\neq 1}}^{m}(\xi(s)-\lambda_j), \prod_{\substack{j=1\\j\neq 2}}^{m}(\xi(s)-\lambda_j), \ldots, \prod_{\substack{j=1\\j\neq m}}^{m}(\xi(s)-\lambda_j)\right)U^{-1}K\right\rangle = 0, \tag{4.2}$$

where in Eq. (4.2) $\lambda_j, 1 \leqslant j \leqslant m$, are the eigenvalues of A_2.

In Eq. (4.2), from Eq. (3.8)

$$\delta(s) = \det(\xi(s) I_m + A_2)$$

then

$$\delta(s) = \prod_{j=1}^{m} (\xi(s) - \lambda_j). \tag{4.3}$$

If now from Eq. (4.1)

$$HU = (f_1, f_2, \ldots, f_m) \tag{4.4}$$

and since

$$U^{-1} = V$$

then

$$VK = (g_1, g_2, \ldots, g_m)^{\mathrm{t}} \tag{4.5}$$

enabling Eq. (4.2) to be written in the form

$$\delta(s) + \left\langle f \left(\prod_{\substack{j=1\\ j\neq 1}}^{m} (\xi(s) - \lambda_j), \prod_{\substack{j=1\\ j\neq 2}}^{m} (\xi(s) - \lambda_j), \ \ldots, \prod_{\substack{j=1\\ j\neq m}}^{m} (\xi(s) - \lambda_j) \right) g \right\rangle = 0, \tag{4.6}$$

which in matrix form expands to

$$(1, \xi(s), \ \xi(s)^2, \ldots, \xi(s)^{m-1}) E_m(\lambda) (f_1 g_1, f_2 g_2, \ldots, f_m g_m)^{\mathrm{t}} = -\left(\prod_{j=1}^{m} (\xi(s) - \lambda_j \right), \tag{4.7}$$

where in Eq. (4.7) $E_m(\lambda)$ is a matrix of elementary functions in λ_j, $1 \leqslant j \leqslant m$, which has the form

$$E_m(\lambda) = \begin{bmatrix} (-1)^{m-1} \prod\limits_{\substack{j=1\\ j\neq 1}}^{m} \lambda_j & (-1)^{m-1} \prod\limits_{\substack{j=1\\ j\neq 2}}^{m} \lambda_j & \cdots & (-1)^{m-1} \prod\limits_{\substack{j=1\\ j\neq m}}^{m} \lambda_j \\ (-1)^{m-2} \sum\limits_{k=1}^{m} \prod\limits_{\substack{j=1\\ j\neq 1,k+1}}^{m-1} \lambda_j & (-1)^{m-2} \sum\limits_{k=1}^{m-1} \prod\limits_{\substack{j\neq 1\\ j\neq 2,k+1}}^{m-1} \lambda_j & \cdots & \\ (-1)^{m-3} \sum\limits_{k=1}^{m+1} \prod\limits_{j\neq 1,k+1}^{m-2} \lambda_j & \vdots & & \\ (-1) \sum\limits_{\substack{j=1\\ j\neq 1}}^{m} \lambda_j & (-1) \sum\limits_{\substack{j=1\\ j\neq 2}}^{m} \lambda_j & \cdots & (-1) \sum\limits_{\substack{j=1\\ j\neq m}}^{m} \lambda_j \\ 1 & 1 & & 1 \end{bmatrix}. \tag{4.8}$$

If for example $m = 3$ then Eq. (4.8) becomes

$$E_3(\lambda) = \begin{bmatrix} \lambda_2 \lambda_3 & \lambda_1 \lambda_3 & \lambda_1 \lambda_2 \\ -(\lambda_2 + \lambda_3) & -(\lambda_1 + \lambda_3) & -(\lambda_1 + \lambda_2) \\ 1 & 1 & 1 \end{bmatrix}.$$

5. Selection of the natural frequencies

If in Eq. (4.7) the substitution of

$$\xi(s) = \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_m$$

is made then these values define the system natural frequencies from which the damped natural frequencies of oscillation can be calculated. This is apparent from Eq. (2.7) where equating $\xi(s)$ to $\gamma_j,\ 1 \leqslant j \leqslant m$, gives

$$s^2 + 2v\omega_{\mathrm{n}}s - \gamma_j = 0,$$

where

$$\omega_{\mathrm{n}}^2 = -\gamma_j.$$

Introducing this substitution into Eq. (4.7) results in

$$W_m(\gamma)E_m(\lambda)(f_1g_1, f_2g_2, \ldots, f_mg_m)^{\mathrm{t}} = P_m(\gamma, \lambda), \tag{5.1}$$

where in Eq. (5.1) the $(m \times m)$ matrices are:

$$W_m(\gamma) = \begin{bmatrix} 1 & \gamma_1 & \gamma_1^2 & \cdots & \gamma_1^{m-1} \\ 1 & \gamma_2 & \gamma_2^2 & \cdots & \gamma_2^{m-1} \\ \vdots & & & & \\ 1 & \gamma_m & \gamma_m^2 & \cdots & \gamma_m^{m-1} \end{bmatrix}, \tag{5.2}$$

$$P_m(\gamma, \lambda) = -\begin{bmatrix} (\gamma_1 - \lambda_1)(\gamma_1 - \lambda_2) & \cdots & (\gamma_1 - \lambda_m) \\ (\gamma_2 - \lambda_1)(\gamma_2 - \lambda_2) & \cdots & (\gamma_2 - \lambda_m) \\ \vdots & & \\ (\gamma_m - \lambda_1)(\gamma_m - \lambda_2) & \cdots & (\gamma_m - \lambda_m) \end{bmatrix} \tag{5.3}$$

and $E_m(\lambda)$ is as defined earlier in Eq. (4.8). Importantly the matrix given in Eq. (5.2) has an explicit inverse and $W_m(\gamma)$ is known as the Wronskian.

Essentially the complete solution of Eq. (5.1) is available here providing, as shown by Wonham [6],

$$A_2,\ \ KH$$

in Eq. (3.1) form a controllable pair. Under these circumstances there is no singularity cancellation and the inversion is cyclic.

The solution to Eq. (5.1) is therefore

$$(f_1g_1,\ f_2g_2, \ldots, f_mg_m)^{\mathrm{t}} = E_m(\lambda)^{-1}W_m(\gamma)^{-1}P_m(\gamma, \lambda), \tag{5.4}$$

where in Eq. (5.4)

$$W_m(\gamma)^{-1} = E_m(\gamma)D_m(\gamma), \tag{5.5}$$

where $E_m(\gamma)$ is as defined in Eq. (4.8) with λ_j replaced by $\gamma_j,\ 1 \leqslant j \leqslant m$ and

$$D_m(\gamma) = \mathrm{Diag}\left(1\Big/\prod_{\substack{j=1\\ j\neq 1}}^{m}(\gamma_1 - \gamma_j),\ 1\Big/\prod_{\substack{j=1\\ j\neq 2}}^{m}(\gamma_2 - \gamma_j), \ldots, 1\Big/\prod_{\substack{j=1\\ j\neq m}}^{m}(\gamma_m - \gamma_j) \right). \tag{5.6}$$

Also from Eq. (5.5) it is evident that

$$E_m(\lambda)^{-1} = D_m(\lambda)W_m(\lambda), \tag{5.7}$$

so that from Eq. (5.7), under cyclic conditions, an explicit inverse for $E_m(\lambda)$ also exists.

6. Evaluation of the feedback law

To evaluate the feedback gains Eq. (5.4) must be employed using the expression for $E_m(\lambda)^{-1}$ and $W_m(\gamma)^{-1}$ given in Eqs. (5.5)–(5.7). This results in

$$(f_1g_1, f_2g_2, \ldots, f_mg_m)^{\mathrm{t}} = D_m(\lambda)W_m(\lambda)E_m(\gamma)D_m(\gamma)P_m(\gamma,\lambda). \tag{6.1}$$

There is no need to evaluate Eq. (6.1) in general terms since the final algorithm is in fact quite simple and generalisable for any value of $m > 0$. However, the "mechanics" of the operation are interesting and following through the procedure for $m = 4$ has been included here by way of demonstration. The intricacies of the algebra remain impressive even when the end result is known.

Multiplying gives for example for the central terms in Eq. (6.1)

$$W_4(\lambda)E_4(\gamma) = \begin{bmatrix} \prod\limits_{\substack{j=1\\ j\neq 1}}^{4}(\lambda_1-\gamma_j) & \prod\limits_{\substack{j=1\\ j\neq 2}}^{4}(\lambda_1-\gamma_j) & \prod\limits_{\substack{j=1\\ j\neq 3}}^{4}(\lambda_1-\gamma_j) & \prod\limits_{\substack{j=1\\ j\neq 4}}^{4}(\lambda_1-\gamma_j) \\ \prod\limits_{\substack{j=1\\ j\neq 1}}^{4}(\lambda_2-\gamma_j) & \prod\limits_{\substack{j=1\\ j\neq 2}}^{4}(\lambda_2-\gamma_j) & \cdots & \vdots \\ \vdots & \vdots & & \\ \prod\limits_{\substack{j=1\\ j\neq 1}}^{4}(\lambda_4-\gamma_j) & & & \prod\limits_{\substack{j=1\\ j\neq 4}}^{4}(\lambda_4-\gamma_j) \end{bmatrix}. \tag{6.2}$$

Pre and post multiplying Eq. (6.2) by the diagonal matrices $D_4(\lambda)$ and $D_4(\gamma)P_4(\gamma,\lambda)$, respectively, as in Eq. (6.1) leads to

$$(g_1f_1,\ g_2f_2,\ g_3f_3,\ g_4f_4)^{\mathrm{t}} = \begin{bmatrix} \dfrac{\prod_{j=1}^{4}(\lambda_1-\gamma_j)}{\prod_{i=1,i\neq 1}^{4}(\lambda_1-\lambda_i)}, & \dfrac{\prod_{j=1}^{4}(\lambda_2-\gamma_j)}{\prod_{i=1,i\neq 2}^{4}(\lambda_2-\lambda_i)}, & \dfrac{\prod_{j=1}^{4}(\lambda_3-\gamma_j)}{\prod_{i=1,i\neq 3}^{4}(\lambda_3-\lambda_i)}, & \dfrac{\prod_{j=1}^{4}(\lambda_4-\gamma_j)}{\prod_{i=1,i\neq 4}^{4}(\lambda_4-\lambda_i)} \end{bmatrix}^{\mathrm{t}} \times \left[-\sum_{k=1}^{4}\prod_{j=1\ j=i}^{4}\left(\frac{\gamma_k-\lambda_j}{\gamma_k-\gamma_j}\right)\right]. \tag{6.3}$$

The scalar multiplier in Eq. (6.3) is unity. Consequently, the general form for the feedback law again takes the form shown by Crossley and Porter [9], even though output feedback is employed, which is

$$f_k = \prod_{j=1}^{m}(\lambda_k-\gamma_j) \Bigg/ \left(g_k\left(\prod_{\substack{i=1\\ i\neq k}}^{m}(\lambda_k-\lambda_i)\right)\right), \tag{6.4}$$

where in Eq. (6.4)

$$g_k = kv_{k_1}$$

and v_{k_1} is the first element in the kth row of V. From Eq. (6.4) it is apparent that since all the $\lambda_j,\ 1 \leqslant j \leqslant m$ are known from Eq. (3.9) then the $\gamma_j,\ 1 \leqslant j \leqslant m$ can be any arbitrary set of real numbers which for stability of operation must be negative.

7. Rotating systems

The control problem for rotating systems can be simply illustrated by considering the representation shown in Fig. 1. Here, the open loop equations of motion with negligible damping for changes in $T(t)$, $T_1(t)$, θ_1 and θ_2 are:

$$J_1 D^2\theta_1 + k(\theta_1 - \theta_2) = T(t) - T_1(t), \qquad -k(\theta_1 - \theta_2) + J_2 D^2\theta_2 = -T_2(t) \tag{7.1}$$

and a control strategy to provide a closed loop damping ratios of 0.28 and 0.5 and natural frequencies of 1 and $\sqrt{3}$ rad/s, respectively, under position regulation conditions is required. The driving torque in Eqs. (7.1) is $T(t)$ and the retarding torques from the brake units are $T_1(t)$ and $T_2(t)$.

Following Laplace transformation with zero initial conditions Eq. (7.1) becomes:

$$\left[\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} s^2 + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\right] \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = \begin{bmatrix} T(s) \\ 0 \end{bmatrix} - \begin{bmatrix} T_1(s) \\ T_2(s) \end{bmatrix}. \tag{7.2}$$

If in Eq. (7.2)

$$\begin{bmatrix} T(s) \\ 0 \end{bmatrix} = \overline{K}(r(s) - H\theta(s)) \tag{7.3}$$

and

$$\begin{bmatrix} T_1(s) \\ T_2(s) \end{bmatrix} = s\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}\theta(s),$$

where $\theta(s) = (\theta_1(s), \theta_2(s))^{\mathrm{t}}$ and $\overline{K} = (\bar{k},\ 0)^{\mathrm{t}}$.

Following the substitution for the control law and arranging in monic form Eq. (7.3) becomes

$$\left[(s^2 + 2v\omega_{\mathrm{n}} s)I_2 + \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}^{-1} \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\right] \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = K(r(s) - H\omega(s)), \tag{7.4}$$

where in Eq. (7.4) to equalise the transient decay rates on both load inertias

$$c_1/J_1 = c_2/J_2 = 2v\omega_{\mathrm{n}},$$

with $K = (\bar{k}/J_1, 0)^{\mathrm{t}},\ H = (h_1, h_2),\ r(s)$ is the reference set point, v the damping ratio and ω_{n} the natural frequency.

Eq. (7.4) becomes

$$(I_2(s^2 + 2v\omega_{\mathrm{n}} s) + A_2 + KH)\theta(s) = Kr(s), \tag{7.5}$$

so that Eq. (7.5) can now be written as

$$(I_2\xi(s) + A_2)(I_2 + (I\xi(s) + A_2)^{-1}KH)\theta(s) = Kr(s). \tag{7.6}$$

In Eq. (7.6), as in Eq. (2.7)

$$\xi(s) = s^2 + 2v\omega_n s,$$

where $2v\omega_n$ is the coefficient of the derivative feedback term and

$$A_2 = U\Lambda U^{-1}.$$

Eq. (7.6) is the same as Eq. (3.5) and as a consequence the determinant is the one given in Eq. (4.6) with $m = 2$. The gains

$$(f_1g_1, f_2g_2)^t \tag{7.7}$$

may now be calculated from Eq. (6.4) where

$$g_1 = v_{11}, \quad g_2 = \ v_{21}$$

and

$$H = (f_1, f_2)U^{-1}.$$

For purposes of illustration let

$$J_1 = J_2 = 100 \text{ kg m}^2 \quad \text{and} \quad k = 100 \text{ Nms/rad}.$$

In Eq. (7.3) therefore

$$A_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{7.8}$$

and the eigenvector and eigenvalue matrices for the matrix in Eq. (7.8) are:

$$U = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad U^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}_{/2} \quad \text{and} \quad \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}. \tag{7.9}$$

From Eq. (7.9)

$$\lambda_1 = -2.0 \text{ and } \lambda_2 = 0 \tag{7.10}$$

and since natural frequencies of 1 and $\sqrt{3}$ rad/s are required then

$$\gamma_1 = -1.0 \text{ and } \gamma_2 = -3.0. \tag{7.11}$$

The algorithm generating the control law given by Eq. (6.4) may now be employed so that

$$f_1g_1 = (\lambda_1 - \gamma_1)(\lambda_1 - \gamma_2)/(\lambda_1 - \lambda_2) = 0.5 \tag{7.12}$$

and

$$f_2g_2 = (\lambda_2 - \gamma_1)(\lambda_2 - \gamma_2)/(\lambda_2 - \lambda_1) = 1.5 \tag{7.13}$$

following the substitution of $\lambda_1, \ \lambda_2, \ \gamma_1$ and γ_2 from Eqs. (7.10) and (7.11). Since

$$g_1 = v_{11} = 0.5, \quad g_2 = v_{21} = 0.5$$

then

$$f_1 = 1.0 \text{ and } f_2 = 3.0$$

and

$$(h_1, h_2) = (1.0, 3.0)\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}_{/2} = (2.0, -1.0),$$

which completes the problem.

To validate the result direct substitution of KH into Eq. (7.5) can be undertaken so that this equation becomes

$$\left[I_2(s^2 + 2v\omega_n s) + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 2.0 & 1.0 \\ 0 & 0 \end{bmatrix}\right]\theta(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(s). \tag{7.14}$$

From Eq. (7.14) the solution is

$$\theta(s) = \begin{bmatrix} s^2 + 2v\omega_n s + 3.0 & 0 \\ -1.0 & s^2 + 2v\omega_n s + 1.0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(s). \tag{7.15}$$

From Eq. (7.15)

$$\theta(s) = \frac{\begin{bmatrix} s^2 + 2v\omega_n s + 1.0 & 0 \\ 1.0 & s^2 + 2v\omega_n s + 3.0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(s),}{(s^2 + 2v\omega_n s + 3.0)(s^2 + 2v\omega_n s + 1)} \tag{7.16}$$

which provides natural frequencies of 1 and $\sqrt{3}$ rad/s and v may be selected to give the required damping ratio commensurate with the practical constraints of the problems and the dissipation rating of the braking units. Following multiplication and cancellation:

$$\begin{bmatrix} \theta_1(S) \\ \theta_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{(s^2 + 2v\omega_n s + 3)} \\ \dfrac{1}{(s^2 + 2v\omega_n s + 3)(s^2 + 2v\omega_n s + 1)} \end{bmatrix} r(s) \tag{7.17}$$

gives the closed loop transfer function relationship.

The model in transfer function-block form shows the structure of the system and controller. This can be obtained from Eq. (7.2) following inversion yielding:

$$\begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \left[\begin{bmatrix} T(s) \\ 0 \end{bmatrix} - \begin{bmatrix} T_1(s) \\ T_2(s) \end{bmatrix}\right], \tag{7.18}$$

where in Eq. (7.18)

$$g_{11}(s) = (J_2 s^2 + k)/\Delta(s), \quad g_{12}(s) = g_{21} = k/\Delta(s), \quad g_{22}(s) = (J_1 s^2 + k)/\Delta(s)$$

and

$$\Delta(s) = s^2(J_1 J_2 s^2 + k(J_1 + J_2)).$$

If now the controller (Eq. (7.3))

$$\begin{bmatrix} T(s) \\ 0 \end{bmatrix} = \overline{K}(r(s) - H\theta(s))$$

and the dynamometer equation

$$\begin{bmatrix} T_1(s) \\ T_2(s) \end{bmatrix} = s \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \theta(s)$$

are included with Eq. (7.18), the block form shown in Fig. 2 can be constructed.

Using the parameters specified earlier the system simulation is in respect of the model given in Eq. (7.2)

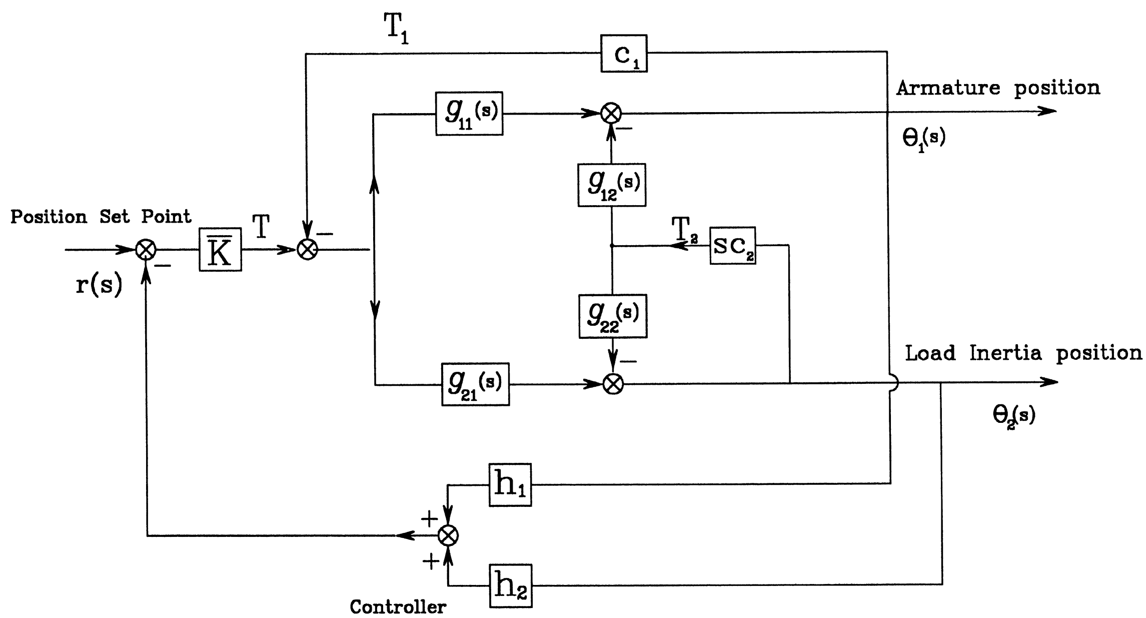

Fig. 2. Block diagram of position control system.

$$\left[\begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} s^2 + \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix}\right] \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = \begin{bmatrix} T(s) \\ 0 \end{bmatrix} - \begin{bmatrix} T_1(s) \\ T_2(s) \end{bmatrix}, \tag{7.19}$$

when employing the feedback, from Eq. (7.4), of

$$\begin{bmatrix} T(s) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left[r(s) \quad - \quad (2.0, 1.0) \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} \right]$$

and

$$\begin{bmatrix} T_1(s) \\ T_2(s) \end{bmatrix} = s \begin{bmatrix} 100.0 & 0 \\ 0 & 100.0 \end{bmatrix} \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix}. \tag{7.20}$$

Since

$$2v\omega_n = c_1/J_1 = c_2/J_2 = 1.0$$

then the damped natural frequencies of the closed loop system can be calculated from

$$\omega_d = \omega_n(1 - v^2)^{1/2},$$

since

$$\omega_n = \sqrt{3} \text{ and } 1 \text{ rad/s},$$

then

$$v_1 = 0.288 \text{ and } v_2 = 0.50 \quad \text{and} \quad \omega_d = 1.658 \text{ and } 0.866 \text{ rad/s}.$$

The simulation, represented by the diagram in Fig. 4, should therefore produce the closed loop response, also predicted by Eq. (7.17) which now becomes

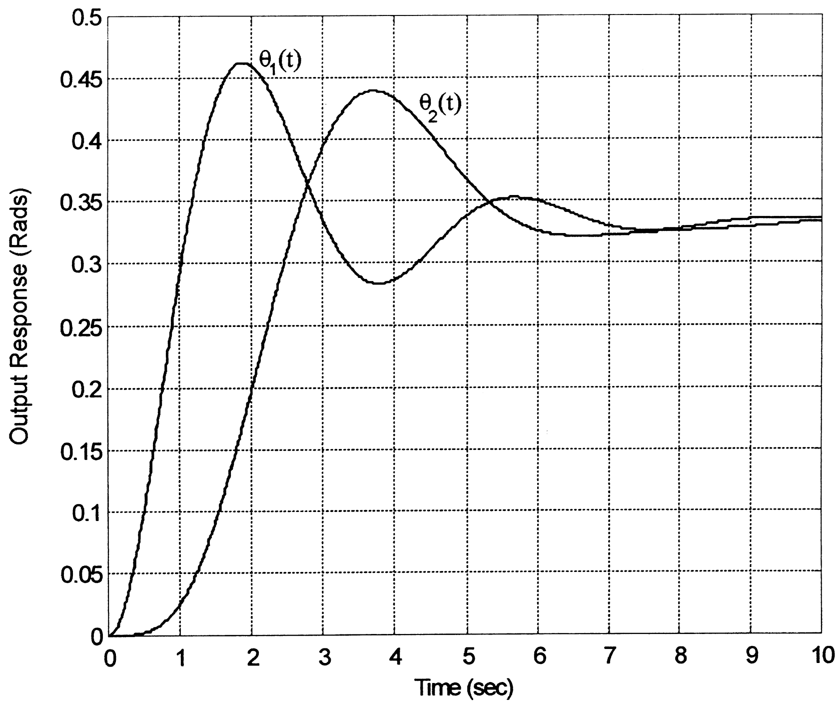

Fig. 3. Output response following a unit step change on the reference input.

$$\begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = \begin{bmatrix} 1/(s^2+s+3) \\ 1/(s^2+s+3)(s^2+s+1) \end{bmatrix} r(s). \tag{7.21}$$

For a unit step change in $r(s)$ Eq. (7.21) produces the characteristics shown in Fig. 3. These curves overlap those from the simulation of Eqs. (7.19) and (7.20) with the same unit step change applied, as would be expected, validating the calculated feedback strategy thereby.

8. Conclusion

In this contribution the arbitrary selection of the exponential decay rates and natural frequency of matrix quadratic models comprising a single, controllable input and any number of outputs was considered.

A new derivation leading to the main control algorithm, enabling the existing and desired natural frequencies to be employed directly in the feedback gain calculation, was presented.

Use of the impedance rather than the admittance equations with state observers was advocated providing thereby a simple, attractive feedback structure. Only m rather than $2m$ feedback measurements are required when using this route and natural outputs rather than state measurements are all that are needed.

In the problem employed to demonstrate the numerical procedures position regulation was required and damping by feedback means was also introduced. Only proportional and tachogenerator feedback was required in this application.

Computation of the required gains and the application of the resulting feedback scheme, in accordance with the theoretical development presented, achieved this requirement providing thereby the closed loop transfer function denominators specified. This was the targeted result in

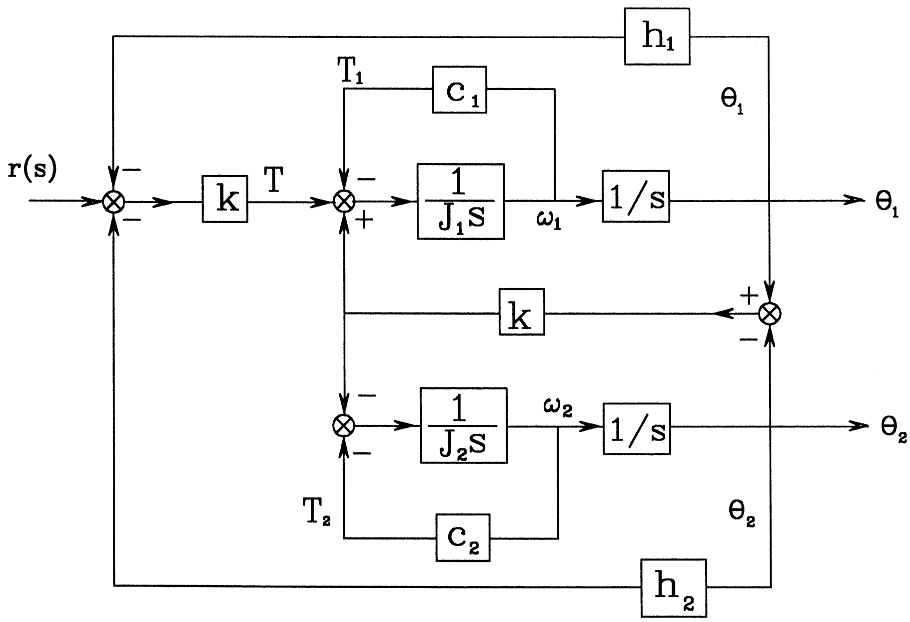

Fig. 4. Closed loop drive block diagram.

the arrangements under consideration involving mechanical system frequency and position regulation, respectively.

In the application considered the outputs from the system are easy to detect and thereafter gain adjustment to give the relevant signals can be simply implemented. The final validation by way of computing the closed transfer functions was for purposes of illustration only. There is no need to proceed beyond the derivation of the impedance, matrix quadratic, in the quest for a suitable controller.

Simulation is of course mandatory in such exercises, where with industrial problems, see for example [2], there may be many outputs to consider. Under these circumstances there may also be numerical computing difficulties to establish the transfer function matrix with the required degree of accuracy.

Here the simulation task is quite easy. As indicated, the impedance equations and feedback signals were digitally integrated, using the representation in Fig. 4 to yield the two outputs, given in Fig. 3, following a step change of unity on the reference input.

It is evident from these figures that the decay rate envelope of $\exp(-v\omega_n t)$ has been enforced and with:

$$v\omega_n = 0.5, \quad -\gamma_1 = \sqrt{3}\,\text{rad/s} \quad \text{and} \quad -\gamma_2 = 1\,\text{rad/s}$$

yielding damped natural frequencies of 1.658 and 0.866 rad/s, with damping ratios of 0.288 and 0.50, respectively.

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